Testing General Relativity with LAGEOS, LAGEOS II and Ajisai laser-ranged satellites

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Abstract

The accuracy reached in the past few years by Satellite Laser Ranging (SLR) allows for measuring even tiny features of the Earth's gravitational field predicted by Einstein's General Relativity by means of artificial satellites. The gravitomagnetic dragging of the orbit of a test body is currently under measurement by analyzing a suitable combination of the orbital residuals of LAGEOS and LAGEOS II. The lower bound of the error in this experiment amount to 12.92%. It is due to the mismodeling in the even zonal harmonics of the geopotential which are the most important sources of systematic error. A similar approach could be used in order to measure the relativistic gravitoelectric pericenter shift in the field of the Earth with a lower bound of the systematic relative error of $6.59 \cdot 10^{-3}$ due to the even zonal harmonics as well. The inclusion of the ranging data to the Japanese passive geodetic satellite Ajisai would improve such limits to 10.78% and $8.1 \cdot 10^{-4}$ respectively and would allow to improve the accuracy in the determination of the PPN parameters β and γ.

1. Introduction

General Relativity, in its slow-motion and weak-field approximation, predicts that a central spherically symmetric body, both if it rotates and if it is static, induces on the Keplerian orbital elements (Sterne, 1960) of a test body orbiting it certain small effects which are unknown in Newtonian classical mechanics (Ciufolini and Wheeler, 1995).

The most famous of them is the well known gravitoelectric precession of the pericenter ω generated by the Schwarzschild's metric of a central, static spherical body (Ciufolini and Wheeler, 1995). It amounts to:

$$\dot{\omega}_{\mathrm{GR}} = \frac{3nGM}{c^2 a(1 - e^2)}, \qquad (1)$$

in which G is the Newtonian gravitational constant, c is the speed of light in vacuum, M is the mass of the central object, a and e are the semimajor axis and the eccentricity, respectively, of the orbit of the test body and $n = \sqrt{GM/a^3}$ is its mean motion. Such effect was detected, up to now, by measuring with the radar ranging technique the Mercury's

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perihelion advance in the field of the Sun at an accuracy level of the order of 1% by including certain sources of systematic errors (Shapiro et al., 1972; Shapiro, 1990). It constitutes one of the classical tests of General Relativity; unfortunately, its interpretation is affected by the currently still existing uncertainty in the quadrupole mass moment $J_\odot$ of the Sun (Ciufolini and Wheeler, 1995; Pireaux et al., 2001). A first attempt to measure it in the field of the Earth by analyzing the laser ranging data to LAGEOS SLR satellite was reported in (Ciufolini and Matzner, 1992), but the accuracy was 20% only. For LAGEOS and LAGEOS II the relativistic gravitoelectric perigee precession due to the Earth's mass amount to 3312.35 and 3387.46 milliarcseconds per year (mas/y in the following) respectively.

Much more smaller is the effect of the proper angular momentum J of the central spherical body on the node Ω and the perigee ω of the orbiting test body. Such feature, derived from the Einstein's equations for the first time by Lense and Thirring in 1918 (Lense and Thirring, 1918), is called gravitomagnetism because the general relativistic linearized equations of motion of a test body in the gravitational field of a central rotating body are formally analogous to that governing the motion of an electrically charged particle acted upon by electric and magnetic fields. The gravitomagnetic rates of the node and the perigee of a test body are:

$$\dot\Omega_{\mathrm{LT}} = \frac{2GJ}{c^2 a^3 (1-e^2)^{3/2}}, \tag{2}$$

$$\dot\omega_{\mathrm{LT}} = -\frac{6GJ\cos i}{c^2 a^3 (1-e^2)^{3/2}}, \tag{3}$$

where i is the inclination of the orbital plane to the equatorial plane of the central body. The Lense-Thirring effect was measured for the first time in the field of the Earth by Ciufolini and coworkers in 1998 (Ciufolini et al., 1998; Ciufolini, 2000) with an accuracy of 20% over a time span of 4 years. They analyzed the orbits of LAGEOS and LAGEOS II. For them the gravitomagnetic precessions induced by the rotation of the Earth, whose angular momentum is $J_\oplus = 5.9 \times 10^{33}$ kg m^2 s^{-1}, amount to:

$$\dot\Omega_{\mathrm{LT}}^{\mathrm{LAGEOS}} \simeq 31 \text{ mas/y}, \tag{4}$$

$$\dot\Omega_{\mathrm{LT}}^{\mathrm{LAGEOSII}} \simeq 31.5 \text{ mas/y}, \tag{5}$$

$$\dot\omega_{\mathrm{LT}}^{\mathrm{LAGEOS}} \simeq 31.6 \text{ mas/y}, \tag{6}$$

$$\dot\omega_{\mathrm{LT}}^{\mathrm{LAGEOSII}} \simeq -57 \text{ mas/y}. \tag{7}$$

Here we want to explore the possibility of refining the precision of the measurements of such general relativistic effects in the terrestrial field by using the laser ranged data to Ajisai as well.

The paper is organized as follows: In Section 2 we explain the role of the error budget in such kinds of experiment pointing out what is the major source of systematic uncertainty. In Section 3 we show how the use of data from Ajisai could improve the accuracy of the

measurements of the gravitomagnetic Lense-Thirring effect and of the gravitoelectric perigee shift. Section 4 is devoted to the conclusions.

In Table 1 we summarize the values of the orbital parameters of LAGEOS, LAGEOS II and Ajisai.

Table 1. Orbital parameters of LAGEOS and LAGEOS II and Ajisai satellites.

Orbital Parameter	LAGEOS	LAGEOS II	Ajisai
a	12.27×10^6 m	12.163×10^6 m	7.87×10^6 m
e	0.0045	0.014	0.001
i	110 deg	52.65 deg	50 deg
n	4.65×10^{-4} s^{-1}	4.71×10^{-4} s^{-1}	9.05×10^{-4} s^{-1}

2. Error analysis

When a satellite-based experiment is planned in order to detect such relativistic effects on the orbital elements of artificial satellites in the gravitational field of the Earth the correct evaluation of the error budget is of the utmost importance. Indeed, the accuracy of these measurements is affected mainly by the systematic errors induced by the mismodeling in the various competing forces of the terrestrial environment (Montenbruck and Gill, 2000) which in many cases are quite larger than the relativistic features of interest. For example, in the case of LAGEOS and LAGEOS II, if we analyzed the node or the perigee of a single satellite the error induced on the classical rates of these orbital elements by the bad knowledge of the first two even zonal harmonics J_2, J_4 of the geopotential (Kaula, 1966) would affect sensibly the accuracy of the measurement of the relativistic rates. Regarding the gravitomagnetic shift, such kind of errors would be even larger than the relativistic effect itself for the orbital element considered (Ciufolini, 1996).

In order to reduce the impact of the mismodeled even zonal harmonics of the geopotential on the measurement of the Lense-Thirring effect in 1996 Ciufolini (1996) put forward an interesting strategy based on the use of a suitably weighted combination of the orbital residuals of the nodes of LAGEOS and LAGEOS II and the perigee of LAGEOS II. It is:

$$\delta\dot{\Omega}^{\mathrm{I}} + c_1 \times \delta\dot{\Omega}^{\mathrm{II}} + c_2 \times \delta\dot{\omega}^{\mathrm{II}} = \mu_{\mathrm{LT}} \times 60.2. \tag{8}$$

In it $c_1 = 0.295$, $c_2 = -0.35$, μ_{LT} is the scaling parameter, equal to 1 in General Relativity and 0 in classical mechanics, to be determined with least-squares fits and $\delta\Omega^{\mathrm{I}}$, $\delta\Omega^{\mathrm{II}}$, $\delta\omega^{\mathrm{II}}$ are the orbital residuals, in mas, calculated with the aid of some orbit determination software

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[*] μ_{LT} can be expressed in terms of the PPN parameter γ as $\frac{(1+\gamma)}{2}$, but this fact is not particularly relevant since the relatively large uncertainty in the measurement of μ_{LT} makes it unsuitable in order to constraint seriously γ. In order to meet this requirement other experiments have been proven more useful (Will, 1993).

like UTOPIA or GEODYN, of the nodes of LAGEOS and LAGEOS II and the perigee of
LAGEOS II. General Relativity predicts for eq.(8) a linear trend with a slope of 60.2 mas/y.

Eq.(8) is obtained writing down the equations of the residuals of the precessions of the
nodes of LAGEOS and LAGEOS II and the perigee of LAGEOS II as a non homogeneous
linear algebraic system of three equations in the three unknowns δJ_2, δJ_4 and μ_{LT} and
solving for μ_{LT}, so to cancel out the static and dynamical contributions of the first two even
zonal harmonics of the terrestrial field (Ciufolini, 1996). The so obtained coefficients c_1 and
c_2 depend only on the orbital parameters of LAGEOS and LAGEOS II. However, the other
higher degree even zonal harmonics J_6, J_8, ... do affect the combined residuals. They induce
an aliasing linear trend which cannot be removed from the data: one can only assess as more
reliably as possible the systematic error induced by it on the measurement of μ_{LT}. According
to the covariance matrix of the geopotential coefficients released by the most recent Earth's
gravity model EGM96 (Lemoine et al., 1998), it amounts to $\delta\mu_{zonals} = 12.92\%\mu_{\text{LT}}$. It is
important to stress that it represents the lower bound of the total systematic error.

A similar strategy could be followed for a new, more precise measurement of the gravi-
toelectric perigee shift in the field of the Earth by means of the residual combination (Iorio
et al., 2001):

$$\delta\dot{\omega}^{\text{II}} + k_1 \times \delta\dot{\Omega}^{\text{II}} + k_2 \times \delta\dot{\Omega}^{\text{I}} = \nu_{\text{GR}} \times 3387.46. \tag{9}$$

In it $k_1 = -0.868$, $k_2 = -2.855$, $\nu_{\text{GR}} = (2 + 2\gamma - \beta)/3$ is the scaling parameter, equal to 1 in
General Relativity and 0 in classical mechanics, to be determined with least-squares fits as
well. It is built up with the PPN parameters β and γ in terms of which the alternative metric
theories of gravitation are usually expressed (Will, 1993): in General Relativity $\beta = \gamma = 1$.
General Relativity predicts for eq.(9) a linear trend with a slope of 3387.46 mas/y, entirely
due to the gravitoelectric shift of LAGEOS II perigee. Also this combination allows to cancel
out the effects of J_2 and J_4. In this case the error induced by the remaining mismodeled
zonal harmonics of the static part of the geopotential amounts to $\delta\nu_{zonals} = 0.6\%\nu_{\text{GR}}$.

The other sources of systematic errors are represented by long period harmonic mis-
modeled perturbations like tides, solar radiation pressure, mismodeling in the satellites'
inclinations i, etc. (Ciufolini et al., 1997). Their impact can be reduced by using ade-
quately long time spans T_{obs} or, if their periods P are shorter than T_{obs}, they can be viewed
as empirically fitted quantities and removed from the signal. Moreover, if the time span is an
integer multiple of their periods they average out. Their effect on the combined residuals is
reduced if the coefficients entering the combinations are smaller than unity, as it is in these
cases. For a recent analysis of their impact on the gravitomagnetic LAGEOS experiment
see (Iorio and Pavlis, 2001).

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[*]Since the observable for the perigee is $ea\dot{\omega}$ and $e_{\text{LAGEOS}} = 0.0045$, the perigee of LAGEOS turns out to
be difficult to be measured accurately enough to detect its gravitomagnetic shift.

3. The role of Ajisai

For a given residual combination, the systematic error induced by aliasing linear trends, like that produced by the mismodeled static part of the geopotential, neither reduces as the time goes along nor can be eliminated by handling suitably the data. A way to reduce such error could be the inclusion in the residual combinations of more orbital elements so to cancel out other higher degree even zonal harmonics J_{2n}, $n \geq 3$. If possible, the choice of the additional orbital elements should be done in order to enhance the slope of the relativistic trends; moreover, they should not introduce too large additional perturbations. This goal restricts our choice to the node and the perigee of the other existing passive geodetic laser-tracked satellites Ajisai, Starlette, Stella, Westpac-1, Etalon-1 and Etalon-2. For a previous analysis of their possible use in measuring the gravitomagnetic effect see (Casotto et al., 1990). The perigee is a very "dirty" element due to the large number of gravitational and nongravitational perturbations affecting it and it could be measured with low accuracy because the orbits of most of the geodetic satellites are even less elliptical than that of LAGEOS and LAGEOS II. So, we focus our attention on the node which is affected by the gravitomagnetic force (but not by the relativistic gravitoelectric one), is one of the most accurately measurable orbital elements and is less sensitive to the orbital perturbations of the terrestrial environment than the perigee.

It turns out that the contributions of the mismodeled higher degree even zonal harmonics to the classical nodal rates of Starlette, Stella and Westpac-1, due to their lower altitude, are too large and would raise the error in the relativistic trends. About the Lense-Thirring effect, in Table 2 we report the most favourable alternative combinations including the perigee of LAGEOS II and in Table 3 the related systematic errors due to the remaining zonal harmonics are quoted. It is assumed that $\delta\dot{\Omega}^{\mathrm{I}}$ is always present multiplied by 1. The slopes of the gravitomagnetic trends, in mas/y, are denoted by x_{LT} and $\delta\mu_{\mathrm{LT}}$ represents the percent systematic error due to the even zonal harmonics of the geopotential. The combination by Ciufolini is denoted by C. In Table 2 Aji=Ajisai, Str=Starlette, Stl=Stella and WS=Westpac-1.

From Table 3 it can be noticed that, with the exception of combination 1, all the combinations other than that by Ciufolini would be not competitive with it because they would be affected by larger systematic errors. Etalon-1 and Etalon-2 turn out to be unsuitable because, if included in the combinations, from one hand they would greatly reduce the error of the static part of the geopotential, but from the other hand they would induce very long period nonzonal perturbations of tidal origin which would affect the combined residuals and resemble linear trends over time spans of few years. Ajisai, in this optics, lies at an intermediate stage. Its gravitomagnetic nodal precession amounts to 115.6 mas/y. By using its node in order to cancel out the effect of J_6 we obtain for the gravitomagnetic experiment

Table 2. Alternative combinations

	Ω^{II}	Ω^{Aji}	Ω^{Str}	Ω^{Stl}	Ω^{WS}	ω^{II}
C	x	0	0	0	0	x
1	x	x	0	0	0	x
2	x	x	0	x	0	x
3	x	x	x	x	0	x
4	x	0	x	x	0	x
5	x	0	x	0	x	x
6	x	x	0	0	x	x
7	x	x	x	0	x	x
8	x	0	x	0	0	x

Table 3. Alternative combinations: numerical values

	c_1	c_2	c_3	c_4	c_5	x_{LT} (mas/y)	$\delta\mu_{\mathrm{LT}}$ (%)
C	0.295	-0.35	0	0	0	60.2	12.92
1	0.443	-0.0275	-0.341	0	0	61.26	10.78
2	0.405	-0.017	-0.0087	-0.306	0	57.85	13.6
3	0.407	-0.0109	0.001	-0.0088	-0.305	57.85	13.65
4	0.38	-0.0096	-0.0085	-0.307	0	57.85	14.75
5	0.381	-0.0097	-0.00904	-0.308	0	57.82	16.46
6	0.405	-0.017	-0.0093	-0.306	0	57.82	16.5
7	0.407	-0.018	0.00064	-0.0093	-0.306	57.81	16.5
8	0.404	-0.015	-0.343	0	0	61.11	16.74

the combination 1:

$$\delta\dot{\Omega}^{I} + c_1 \times \delta\dot{\Omega}^{II} + c_2 \times \delta\dot{\Omega}^{Aj} + c_3 \times \delta\dot{\omega}^{II} = \mu_{\mathrm{LT}} \times 61.26. \qquad (10)$$

In it $c_1 = 0.443$, $c_2 = -0.0275$ and $c_3 = -0.341$. Regarding the gravitoelectric experiment we have:

$$\delta\dot{\omega}^{II} + k_1 \times \delta\dot{\Omega}^{II} + k_2 \times \delta\dot{\Omega}^{I} + k_3 \times \delta\dot{\Omega}^{Aj} + k_4 \times \delta\dot{\omega}^{I} = \nu_{\mathrm{GR}} \times 7928.21. \qquad (11)$$

In it $k_1 = -1.962$, $k_2 = -3.693$, $k_3 = 0.036$ and $k_4 = 1.370$. The coefficients of Ajisai are smaller than that of LAGEOS and LAGEOS II due to its different orbital parameters. The inclusion of the residuals of Ajisai's node, in fact, would introduce an improvement in reducing the systematic errors due to the geopotential: indeed, according to EGM96 model, they amount to $\delta\mu_{zonals} = 10.78\%\mu_{\mathrm{LT}}$ and $\delta\nu_{zonals} = 0.08\%\nu_{\mathrm{GR}}$. Furthermore, the coefficients with which it would enter the modified combinations are sufficiently small to depress the impact of the other gravitational and nongravitational perturbations (Sengoku et al., 1995; 1997) acting on it. This is especially true for the Lense-Thirring effect: indeed,

in eq.(10) the coefficients of the elements of LAGEOS and LAGEOS II are close to those of eq.(8), while the coefficient of Ajisai is of the order of 10^{-2} only; in particular, the perigee of LAGEOS II, which is the major source of mismodeled perturbations, is weighted at the same level. This means that to the slight improvement in the error due to the geopotential it should not correspond a worsening of the time-varying part of the error budget which should be remain almost the same as in the current LAGEOS experiment.

4. Conclusions

The use of suitable combinations of the orbital residuals of the nodes of LAGEOS, LAGEOS II and Ajisai and the perigee of LAGEOS II would improve the precision of the measurements of certain subtle general relativistic effects in the gravitational field of the Earth. In particular, regarding the Lense-Thirring experiment, currently performed by analyzing the data of the two LAGEOS and LAGEOS II only, the systematic error induced by the even zonal harmonics of the geopotential, which is the major source of uncertainty, would reduce from the present 12.92% to 10.78% with Ajisai's node. Concerning the proposed measurement of the relativistic gravitoelectric pericenter shift, the error induced by the geopotential would pass from 0.6% to 0.08% by including also the perigee of LAGEOS and the node of Ajisai.

Regarding the improvements which could be obtained in the accuracy of our knowledge of the PPN parameters β and γ, if from one hand even a refined version of the Lense-Thirring experiment would not be particularly useful in constraining effectively γ through the measurement of μ_{LT}, from the other hand the proposed gravitoelectric experiment (Iorio et al., 2001) could be able to obtain interesting results raising the accuracy on γ and β to the $10^{-3} - 10^{-4}$ level and providing us with an independent measurement of them in the field of the Earth with laser-ranging.

It must be pointed out that both such estimates will greatly improve in the near future when the new data on the terrestrial gravitational field will be released by the CHAMP and GRACE missions.

Concerning the time-dependent part of the error budget, a detailed analysis of the impact of the harmonic perturbations of gravitational and nongravitational origin on the node of Ajisai in the context of such relativistic measurements would be required.

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